Proposal of Integral-Differential Equation Model for Prevalence of Substance Use

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Abstract

Objective: Prevalence of substance use is among the problems that have increasingly grown around the world and affect the addicted individuals' social interactions with others. The aim of this paper was to introduce an integral-differential equation model for the prevalence of substance abuse among the people of a supposed population. **Method:** The connection structures of the discrete systems and networks that exist among the members of a supposed collection can end in a discrete networking model and, finally, in a model of differential-integral equation in continuous mode. In this research, the level of drug users' purity and health has been investigated by considering peer influence. **Results:** In this study, in addition to the proposal of simple discrete models for the prevalence of substance abuse, a new alternative model along with an integral-differential equation model is introduced that avoids the inertia existing in the previous models related to the prevalence of substance abuse problem. **Conclusion:** Some analyses were conducted on discrete network models for the prevalence of substance abuse in a supposed population and the mathematical equation was proposed for the diffusion of substance abuse by considering individuals' resilience and peer influence.

Keywords: mathematical model, drug dependence, prevalence

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Introduction

Based on the nature of science, most of the natural events and accidents that we experience in reality can be expressed in the language of mathematics. This language conversion helps to use the unlimited tools available in mathematical science and, consequently, draw conclusions as a result of the conduct of the analysis and programming. Drug abuse is one of the most rapidly growing problems, and its prevalence among societies has affected other people, especially healthy people more than ever. Since the consumption of various types of drugs is associated with mental disorders, this can be the basis for the incidence of many accidents and crimes, such as traffic accidents, conflicts, and other social harms in the country. The level of drug use is not the same among the members of any given society and depends on factors, such as individuals' flexibility and addicts' impact on other people in the community. In this research, it was attempted to present an integral-differential equation model by means of discrete network systems for investigating the prevalence of drug addiction. The reason for the selection of the differential-integral equation model is that it has been proved in recent years that differential equations and integral-differential have been frequently used in the mathematical modeling of the prevalence of infectious and viral diseases that are epidemiologically and asymptomatically similar to addiction (Medlock, & Kot, 2003). Since addiction has a reversible effect and the affected person will not be safe even after recovery, this reversibility feature for the diseases similar to addiction can only be modeled by the differential-integral equation models (Van den driessche, & Zou, 2007). It should be noted that while the consumption of narcotics and the individuals' dependence on the disease are taken into consideration, the prevalence and epidemiology of this disease are clearly different from the prevalence of infectious and viral diseases, such as measles, influenza, etc. So far, no similar mathematic model has been proposed in relation to the prevalence of drug use. However, similar studies have been carried out in relation to the prevalence of alcohol consumption through a similar way in the modeling (Braun, Wilson, Pelesko, Buchanan, & Gleeson, 2006; Wilson, Buchanan, Gleeson, & Braun, 2004; French, Teymuroglu, Lewis, & Braun, 2010). In the mentioned sources, a discrete network model along with a stable rate function and calculations has been proposed for the behavior and prevalence of these conditions in a population. Here, in order to analytically test the stability, new simple models have been proposed under the inspiration of a discrete network model and, finally, a differential-integral equation model is introduced. There are a lot of circumstances wherein it has been shown that people from one community can influence the amount of other people's desire and willingness for drug use. In fact, the investigation of previous studies leads one to find that the influence and effect of some individuals from one population on other people represent the same outbreak and prevalence (Funk, Salathe, & Jansen, 2010). For further

reading about the history of the available models in social discussions, readers are hereby referred to the references of this study. Essentially, this study aims to assess the index of drug use by individuals or groups of the community. In this way, the dynamics of a society will be assessed by factors, such as flexibility, the readiness of the people of that society, and their social relationships. The mechanism of these models contains the potential for the simulation of the impact of drug users on non-addicted people (cited in Braun et al., 2006).

Method

Basic differential equation model: The starting point of investigations is the following differential equation:

$$\frac{dv_i}{dt} = v_i(1 - v_i)(n_i - r_i) \quad (1)$$

This equation has a stable rate function and is used in a discrete network system (Braun et al., 2006). In equation (1), the function $v_i = v_i(t)$ models the probability of the incidence of drug-related problems for person i at the moment t (including arrest, car accident, conflict, and arguments). It is assumed that drug-related problems for one person can be considered to be quantitative rather than qualitative. Here, the term "purity" of people or their addiction (addiction to one or more types of drugs) for v_{i} is considered. From now on, this function is introduced as the function of addiction. This function falls within the category of probabilistic functions and takes values between zero and one. So if person i is pure, $v_{\mathrm{i}} \cong 0$; and if person i is addicted$v_{\mathrm{i}} \cong 1$.

Then, $n_i = n_i(t)$ is the impact function, which is used to measure the influence of other people by considering w_{ij} weights.

$$n_i(t) = \sum_j w_{ij} v_j(t),$$

In this case, $w_{\mathrm{ij}} > 0$ is the magnitude of the relationship between person i and j. It is also assumed that the w_{ij}s are normalized ($\sum_j \mathrm{w_{ij}} = 1$ for each i); besides, we can consider the weights w_{ij} symmetric in addition to a particular state ($w_{\mathrm{ij}} = w_{\mathrm{ji}}$ for each j and i). r_{i} is the flexibility of individual i ($0 \leq r_{\mathrm{i}} \leq 1$). Whatever the size of $r_{\mathrm{i}}$is close to number one, person I will be less likely to get addicted. Here, the individuals' flexibility level is shown with constant r. Also, the following initial condition is required for the problem determination and its answer.

$$(2) \quad v_i(0) = g_i$$

Discrete network model: In the mentioned sources, the discrete network model has been considered, which, in some way, models the addiction of person i from the time period m to m + 1 in a large network as follows.

(3) $$v_i^{m+l} - v_i^m = \lambda v_i^m (1 - v_i^m)(n_i^m - r_i) \qquad n_i^m = \sum_j w_{ij} v_j^m.$$

Here, v_i^m is the level of purity and health for individual i at timet_m, which takes values between zero and one; and $\lambda > 0$ is a constant ratio. If we assume that the length of the steps is small and we also re-scale the time, then the differential equation (1) will appear as an approximation out of the complete and large network equation above.

The spread of the disease to the networks has always been one of the most important issues of scientific research in the past, especially when the communication connections and paths are defined between community members (nodes) in the network either randomly or in a predetermined manner. These resources suggest the following relationships:

(4) $$\lim_{m\to\infty} v_i^m = \begin{cases} 1, & \bar{v}_0 > \bar{r} \\ 0, & \bar{v}_0 < \bar{r} \end{cases}$$

Here, $\bar{v}_0$ is the mean and initial index of addiction, and $\bar{r}$is the flexibility of the mean. When$\bar{v}_0 > \bar{r}$, the population will include the primary consumers of narcotics; and when$\bar{v}_0 > \bar{r}$, the population is directed towards purity in an imposed fashion.

In Braun et al.'s work (2006), a therapeutic model has been presented. They have assumed that a small percentage of people in a population with high addiction rates could be treated. These unhealthy people (with high addiction rates) are eliminated for a certain period of time from their populations for the sake of treatment and, then, are returned to a general population with the same attribute of their own addiction, which has now been reduced to half of their level of flexibility. In addition, the numerical calculations lead one to find that if the 7% of the population are strictly treated according to a regular basis, the entire population will get clean and pure. This can be used as a very useful tool in rehabilitating addicted patients.

In the next section, a collection of very simple models derived from the discrete network model will be considered. This set of models contains very simple and important models, including single-variable models, two-patch differential equation model, and the differential-integral model. In the integral-differential equation model, the impact and influence of other people in the population are calculated by a simple mean. Moreover, in the following of introducing the models, a new model will also be introduced as an alternative model using the Heaviside function.

In the last section, the obtained model of the differential-integral equation will be expanded. The main objective here is to bring discrete models into a continuous framework and format. It is fulfilled using the convolution of the two functions v and$w$. The review and testing of this continuous model are more realistic and more practical than the previous discrete model. By means of this

continuous model, the possibility of the conduct of more numerical analysis is given to mathematicians in order to decide on the actual results. It should be noted that our assumption in this paper is that there is a unique solution for the given differential-integral equation, but there is no evidence of its availability here. The interested readers can refer to French et al.' reference (2010) for the proof of the answer.

Instrument

Models inspired by discrete networks: In this section, as it was mentioned, three very discrete models will be presented.

Single variable bitable model: Assume that all people in the population are equal in terms of impact, flexibility, and the value of addiction function (homogeneous population), and these people are in contact with each other with equal communication weights. In this case, the differential equation (1) is converted to the initial value problem with respect to time.

$$\acute{v} = v(1-v)(v-r) \quad , \quad v(0) = v_0$$

It should be noted that zero, one, and r are fixed points and the rate function of $v(1-v)(v-r)$ denotes that if $v_0 < r$, because of the smallness of the additive function v, its flexibility will approach zero (pure mode). In addition, whenever $v_0 > r$, the additive function v will move to a side (addiction mode). These results are exactly compatible with equation (4).

Two-patch model: Another simple model being considered here involves two different populations. At the outset, the hypothesized population is divided into two different groups. The first group includes a large population of addicts with moderate drug addiction, and the second group includes a small population of addicts with high addiction rates who have a high interest in drug use. In this situation, the model will be in the form of an ordinary differential equation in the following state:

(5) $$\acute{v}_1 = v_1(1-v_1)(kv_1 + (1-k)v_2 - r_1), \qquad 0 \le k \le 1,$$

(6) $$\acute{v}_2 = v_2(1-v_2)(kv_1 + (1-k)v_2 - r_2)$$

Where:

(7) $$v_1(0) = v_1^0 \quad , \quad v_2(0) = v_2^0.$$

In this model, we will, in fact, deal with two homogeneous populations.

Suppose that the addiction function v_2 includes a small set of people who are highly eager to consume drugs, with the assumption that v_1 introduces the rest of the population. In order for the drug users' influence to be negligible on a given population, $k \cong 1$.

It should be noted that the following initial conditions are very determinative in the process of work. In reality, the mean value of the initial status of a population is also important at the start of work.

$$v_1(0) = v_1^0 \qquad , \qquad v_2(0) = v_2^0$$

If the majority of the population have had a high rate of dependence on drug use from the outset ($v_1^0 \to 1$) due to the initial conditions, then the entire population will become more addicted with the passage of time. Otherwise, if the initial conditions of the majority of the population are small from the beginning ($v_1^0 \to 0$), then the entire population will be willing to purity and health over time and their readiness and willingness to drug use will be reduced.

Average peer influence model: Here, it is assumed that there are J people that are connected with each other with weights of $w_i = \frac{1}{J}$ $(i = 1, \dots, J)$ and have an equal flexibility and constant r. in this case, the following model of the differential equation will come out:

(8) $$\frac{dv_i}{dt} = v_i(1 - v_i)(n_i - r), \qquad n_i = \frac{1}{J}\sum_{j=1}^{J} v_j, \qquad i = 1, \dots, J.$$

Under the assumption of $v_i = v(x_i, t)$ and with the definition of $x_i = \frac{i}{J}$ in the limit state, the following integral-differential equation on the interval [0, 1] will be considered:

(9) $$\frac{\partial v}{\partial t} = v(1 - v)(n - r)$$

Where

(10) $$n(t) = \int_0^1 v\,(x, t)\,dx$$

The above equation can be considered as an approximation of equation (8). We can immediately conclude the following relationships, which are indeed the advantages of this simple approximation in the continuous space:

(11) $$\lim_{t\to\infty} v(x, t) = \begin{cases} 1 & , \quad \int_0^1 v_0 dx > r, \\ 0 & , \quad \int_0^1 v_0 dx < r, \end{cases}$$

Again, with a little carefulness, it can be observed that the above relations are similar to relations (4).

Now, with respect to the definition of the function $n(\mathrm{t})$ (as a known function), and considering the additive function $v$at a given point of $\bar{x}$, we can obtain the above differential equation in separation:

(12) $$\int_{v_{0(\bar{x})}}^{v(\bar{x},t)} \frac{du}{u(1-u)} = N(t)$$

Where

(13) $$N(t) = \int_0^t (n(s) - r)ds$$

And, immediately, with its solution, we obtain:

(14) $$v(\bar{x}, t) = \left(1 + \frac{1 - v_0(\bar{x})}{v_0(\bar{x})} \mathrm{e}^{-\mathrm{N(t)}}\right)^{-1}$$

If $0 < v_0(\bar{x}) < 1$ for each $\bar{x} \in [0\ ,1]$ in relation (10), then, $0 \leq v \leq 1$.

It should be noted that we obtain a typical differential equation for function n by integrating equation (9) with respect to x:

(15) $$\frac{d}{dt}\left[\int_0^1 v\ dx\right] = \left[\int_0^1 v(1 - v)\, dx\right]\ (n - r)$$

And with the following assumption

(16) $$\beta(t) = \int_0^1 v(1 - v)\, dx.$$

we will have:

(17) $$\frac{d}{dt}(n - r) = \beta(n - r)$$

in such a way that $\beta(t) \geq 0$.

Assume that β is nonnegative, then we can solve the linear differential equation (17) for n-r in terms of β, and we will have:

(18) $$n(t) - r = (n(0) - r)e^{\int_0^t \beta(s)ds}$$

Now, we can arrive at two conclusions:

a) If we assume $n(0) >$ r, then, $n(t) - r \geq n(0) - r$,

then, when $t \to +\infty$, we will have:

$$e^{-N(t)} = e^{-\int_0^t (n(s)-r)ds}$$
$$\leq e^{-(n(0)-r)t} \to 0$$

Therefore, we conclude from (10) that whenever $v(\bar{x}, t) \to 1$ then, $t \to +\infty$.

b) If $n(0) < r$, we will conclude as above that: when $v(\bar{x}, t) \to 0$ then $t \to +\infty$. As a result, relation (11) is proved.

New switch model: Such a model is now recommended that includes a substitution with Heaviside function dependent on the symbol of the function $n_i - r_i$. This model has a long-term behavior and dynamics similar to the double stable machine (3). In this case, we have:

(19) $$\frac{dv_i}{dt} = -H(r_i - n_i)v_i - H(n_i - r_i)(v_i - 1)$$

where

$$n_i(t) = \sum_j w_{ij} v_j(t) \quad (20)$$

And $H = H(s)$ is the Heaviside function.

$$H(s) = \begin{cases} 1 & \text{and } s > 0 \\ 0 & \text{and } s \leq 0 \end{cases}$$

It is notable that if $n_\mathrm{i} < \mathrm{r_i}$, then $\frac{dv_\mathrm{i}}{\mathrm{dt}} = -v_\mathrm{i}$, so $v_\mathrm{i} \to 0$ (purity). And if $n_\mathrm{i} > \mathrm{r_i}$, then $\frac{dv_\mathrm{i}}{\mathrm{dt}} = -(v_\mathrm{i} - 1)$ and $v_\mathrm{i} \to 1$ (addiction). In any case, v_i cannot be greater than one or smaller than zero, which is the same as in the double stable the bitable model.

Finally, since می‌در $H(r_i - n_i) + H(n_i - r_i) = 1$, we can understand that:

$$\frac{dv_i}{dt} = -v_i + H(n_i - r_i) \quad (21)$$

Results

Integral-differential equation model: In this section, a model of differential-integral equation is introduced using the contents of the previous section. In fact, the purpose of presenting this model is to generalize the discrete network model as a continuous model. For the start, we propose the following integral-differential model under the inspiration for the discrete network model.

$$\frac{\partial v}{\partial t} = v(1 - v)(n - r) \quad (22)$$

In this model, $v = v(x, t)$ shows the individuals' level of purity in position x and time t. Similar to what has already been stated, if $v \cong 0$, the intended person is pure and non-addicted; in addition, the addicted people will have an addictive function with a value between zero and one depending on their level of addiction. Obviously, addicts with a high rate of addiction have an addictive function close to 1 ($v \cong 1$).

The function $n = n(x, t)$ shows the influence of other individuals in relation to individuals in the position x and time t, which is equivalent to the convolution of the two functions v and w.

$$n(x, t) = (w * v)(x, t) = \int_{-\infty}^{\infty} w(x - y) v(y, t) dy. \quad (23)$$

Here, $w > 0$ is the weight function or the footprint ($\int_R \mathrm{wds} = 1$). This function $w = w(s)$ imposes a translation invariance to the model, which does not exist in discrete states. We also assume that the individuals' flexibility is also constant ($0 \leq r \leq 1$). To complete the model, we need the initial condition of $v(.,0) = v_{0.}$.

We now consider a community of people who are systematically arranged on the line of real numbers in such a way that the person is placed in the location of $x_i = i\Delta x \quad (0 < \Delta x \ll 1)$. As it was mentioned, it is assumed that the relationships are modeled between the population members with the footprint function of w. With the assumption of $w_{ij} = \Delta x\, w(x_i - x_j)$, we can approximate the function $n_i(t)$in equation (1) by an integral.

$$n_i = \sum_j w_{ij} v_j = \sum_j \Delta x\, w\,(x_i - x_j) v(x_j,.) \cong \int w\,(x_i - y)\, v(y,.)\, dy = n(x_i,.). \tag{24}$$

It is noteworthy that this model, like models (1) and (3), has this feature that when individuals have their own orientation in both paths of purity and addiction, then they will have a lower degree of potential for being affected by other people in the population (affected by the parameter $v(1 - v)$.

Discussion and Conclusion

Based on the proposed model, the analysis, programming, control, and organization of the phenomenon of the prevalence of drug use are mathematically modelable. Based on the behavior of the assumed population, the designed mathematical model designed is used after validation to predict and modify the behavior of the population based on the changes in the parameters. Moreover, with a comprehensive and consistent therapeutic model, we can identify individuals with high addiction potential among a pure and healthy population. With the passage of time, we observe that the healthy people in the given population have been able to have positive effects on addicted people and to reduce their willingness and attachment to drug use. In addition, via the proposed model, the rate of prevalence among individuals in a community can be estimated. This issue can be numerically analyzed with advanced mathematical tools, such as approximation by partial derivatives. It should also be noted that the differential-integral model presented in this paper can be a good start for those interested in mathematical subjects for further analysis. The numerical analysis of the model (24) is proposed to be done with new and more precise methods as new research work so that more accurate analyses can be performed.

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